Coordinate Graph Curriculum *©2015*

Age(s): Upper Elementary & Middle (7-14 years)

Description:

The Work of Circle's Curriculum was designed to introduce students to the basic concepts of the circle and to the more complex concepts of measurement – area, circumference, central angles, and arc lengths – and the calculations of such measurements. The 5-Level Command Cards (with rainbow coloring to indicate the Level) allow for multiple lessons and activities of work. Students begin with the parts of the circle and work their way through the measurements of area, circumference, angles, arcs, sectors and segments.

Materials Included:

- 19"x 19" Wooden Circle Box
- Various wooden circles and circle pieces
- 5-Level Command Cards
  Yellow: Diameter & Radius
  Orange: Circumference
  Red: Area
  Purple: Central Angles & Arcs
  Blue: Arc Length, Sectors & Segments
- 3-piece Nomenclature Cards
- 2- 10cm length green line segments
- 4 – 12 inch length lines
- Protractor
- Curriculum Guide

Presentation I: Introduction to the Circle – Parts of the Circle, Radius, & Diameter, Circumference & Area

Begin the Introduction to the Circle by ***i***ntroducing student(s) to a brief history of the Circle. Please feel free to your own "story"- Montessori style.

A circle isn't something that exists in nature. It isn't something that people discovered like gold or the new lands of America. It is a mental construct, a symbolic representation that was invented much the same as language and the alphabet.

There is no way to be certain, but anthropologists generally agree that the circle was created long before recorded history. It is quite likely that it was drawn by a stick in the sand. With the sun being a constant in early man's existence and the source of all life, it is quite likely that the first circle represented the sun.

Through the years man's understanding of the circle has evolved substantially with Euclidean geometry being its crowning point of technological understanding.

Without the rudimentary understanding of circles, the world would not be anything like it is today. Without circles, there would be no wheel, which is man's crowning achievement dating back to the Neolithic Age (circa 9500 BC).

Besides the wheel, pulleys, gears, ball bearings and a thousand other items we take for granted wouldn't exist. And of course we would never have the pleasure of driving a car, riding a Ferris wheel, or watching the moon landing on our television set.

The circle is the most primitive and rudimentary of all human inventions, and at the same time, the most dynamic. It is the cornerstone in the foundation of science and technology. It is the basic tool of all engineers and designers. It is used by the greatest artists and architects in the history of mankind.

A circle is a simple shape in Euclidean geometry. It is the set of all points in a plane that are at a given distance from a given point, the center; equivalently it is the curve traced out by a point that moves so that its distance from a given point is constant. The distance between any of the points and the center is called the radius.

Natural circles would have been observed, such as the Moon, Sun, and a short plant stalk blowing in the wind on sand, which forms a circle shape in the sand. The circle is the basis for the wheel, which, with related inventions such as gears, makes much of modern machinery possible. In mathematics, the study of the circle has helped inspire the development of geometry, astronomy, and calculus.

Early science, particularly geometry and astrology and astronomy, was connected to the divine for most medieval scholars, and many believed that there was something intrinsically "divine" or "perfect" that could be found in circles.

The first theorems relating to circles are attributed to Thales around 650 BC. Book III of Euclid's Elements deals with properties of circles and problems of inscribing and escribing polygons.

One of the problems of Greek mathematics was the problem of finding a square with the same area as a given circle. Several of the 'famous curves' in this stack were first studied in an attempt to solve this problem. Anaxagoras in 450 BC is the first recorded mathematician to study this problem.

Pi has been known for almost 4000 years—but even if we calculated the number of seconds in those 4000 years and calculated pi to that number of places, we would still only be approximating its actual value. Here's a brief history of finding pi:

The ancient Babylonians calculated the area of a circle by taking 3 times the square of its radius, which gave a value of pi = 3. One Babylonian tablet (ca. 1900–1680 BC) indicates a value of 3.125 for pi, which is a closer approximation.

The Rhind Papyrus (ca.1650 BC) gives us insight into the mathematics of ancient Egypt. The Egyptians calculated the area of a circle by a formula that gave the approximate value of 3.1605 for pi.

The first calculation of pi was done by Archimedes of Syracuse (287–212 BC), one of the greatest mathematicians of the ancient world. Archimedes approximated the area of a circle by using the Pythagorean Theorem to find the areas of two regular polygons: the polygon inscribed within the circle and the polygon within which the circle was circumscribed. Since the actual area of the circle lies between the areas of the inscribed and circumscribed polygons, the areas of the polygons gave upper and lower bounds for the area of the circle. Archimedes knew that he had not found the value of pi but only an approximation within those limits. In this way, Archimedes showed that pi is between 3 1/7 and 3 10/71.

A similar approach was used by Zu Chongzhi (429–501), a brilliant Chinese mathematician and astronomer. Zu Chongzhi would not have been familiar with Archimedes' method—but because his book has been lost, little is known of his work. He calculated the value of the ratio of the circumference of a circle to its diameter to be 355/113. To compute this accuracy for pi, he must have started with an inscribed regular 24,576-gon and performed lengthy calculations involving hundreds of square roots carried out to 9 decimal places.

Mathematicians began using the Greek letter π in the 1700s. Introduced by William Jones in 1706, use of the symbol was popularized by Leonhard Euler, who adopted it in 1737.

In the twentieth century, computers took over the reins of calculation, and this allowed mathematicians to exceed their previous records to get to previously incomprehensible results.

Radius:

The radius of a circle is the length of the line from the center to any point on its edge. The plural form is radii (pronounced "ray-dee-eye").

If you know the diameter

Given the diameter of a circle, the radius is simply half the diameter:

$$radius = \frac{D}{2}$$

where:
D is the diameter of the circle

If you know the circumference

If you know the circumference of a circle, the radius can be found using the formula

$$radius = \frac{C}{2\pi}$$

where:
C is the circumference of the circle
π is Pi, approximately 3.142

If you know the area

If you know the area of a circle, the radius can be found using the formula

$$radius = \sqrt{\frac{A}{\pi}}$$

where:
A is the area of the circle
π is Pi, approximately 3.142

Diameter:

The diameter of a circle is the length of the line through the center and touching two points on its edge.

The diameter is also a chord. A chord is a line that joins any two points on a circle. A diameter is a chord that runs through the center point of the circle. It is the longest possible chord of any circle.

The center of a circle is the midpoint of its diameter. That is, it divides it into two equal parts, each of which is a radius of the circle. The radius is half the diameter.

If you know the radius

Given the radius of a circle, the diameter can be calculated using the formula

$$\text{diameter} = 2R$$

where:
R is the radius of the circle

If you know the circumference

If you know the circumference of a circle, the diameter can be found using the formula

$$\text{diameter} = \frac{C}{\pi}$$

where:
C is the circumference of the circle
π is Pi, approximately 3.142

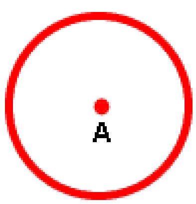
If you know the area

If you know the area of a circle, the diameter can be found using the formula

$$\text{diameter} = \sqrt{\frac{4A}{\pi}}$$

where:
A is the area of the circle
π is Pi, approximately 3.142

<u>Circumference:</u>



A circle is a shape with all points the same distance from the center. It is named by the center. The circle to the left is called circle A since the center is at point A. If you measure the distance around a circle and divide it by the distance across the circle through the center, you will always come close to a particular value, depending upon the accuracy of your measurement. This value is approximately 3.14159265358979323846... We use the Greek letter π *(pronounced Pi)* to represent this value. The number π goes on forever. However, using computers π has been calculated to over 1 trillion digits past the decimal point.

The distance around a circle is called the **circumference**. The distance across a circle through the center is called the **diameter**. π is the ratio of the circumference of a circle to the diameter. Thus, for any circle, if you divide the circumference by the diameter, you get a value close to π. This relationship is expressed in the following formula:

$$\frac{C}{d} = \pi$$

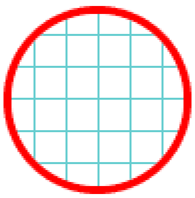
where C is circumference and d is diameter. You can test this formula at home with a round dinner plate. If you measure the circumference and the diameter of the plate and then divide C by d, your quotient should come close to π. Another way to write this

formula is: $C = \pi \cdot d$ where $\cdot$ means multiply. This second formula is commonly used in problems where the diameter is given and the circumference is not known.

The **radius** of a circle is the distance from the center of a circle to any point on the circle. If you place two radii end-to-end in a circle, you would have the same length as one diameter. Thus, the diameter of a circle is twice as long as the radius. This relationship is expressed in the following formula: $d = 2 \cdot r$, where d is the diameter and r is the radius.

Circumference, diameter and radii are measured in linear units, such as inches and centimeters. A circle has many different radii and many different diameters, each passing through the center.

Area:



The area of a circle is the number of square units inside that circle. If each square in the circle to the left has an area of 1 cm^2, you could count the total number of squares to get the area of this circle. Thus, if there were a total of 28.26 squares, the area of this circle would be 28.26 cm^2 However, it is easier to use one of the following formulas: $A = \pi \cdot r^2$ where A is the area, and r is the radius.

Activity Follow-Up & Extensions:

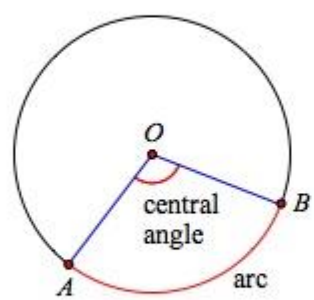
1. Have students read through the Circle Nomenclature Cards and practice matching Nomenclature (3-Part) Cards.
2. Have Students begin with Lesson Cards 1 – 6 (Radius, Diameter, Circumference & Area)
3. Have Students begin Command Cards Level A (yellow (Radius & Diameter), orange (Circumference) & red (Area)).

Notes for instructor:

*****Remind students that 3.14 is an approximation of pi.***

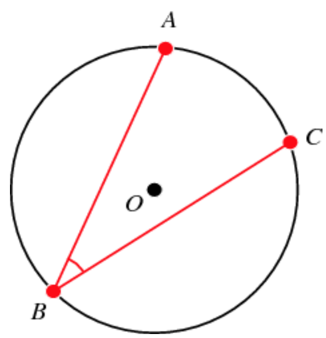
Presentation II: Arcs & Central Angles & Radians

In the previous lesson, you learned how to identify the parts of a circle, measure the diameter and radius of the circle, calculate the perimeter or the circumference of a circle and calculate the area within the circle.



Now we will explore Central Angles, Arcs, & Radians in more detail.

An angle whose vertex is the center of a circle and whose sides pass through a pair of points on the circle is called a ***central angle****. The following symbol, pronounced "theta", is used to represent a central angle:* ***ϑ****.*



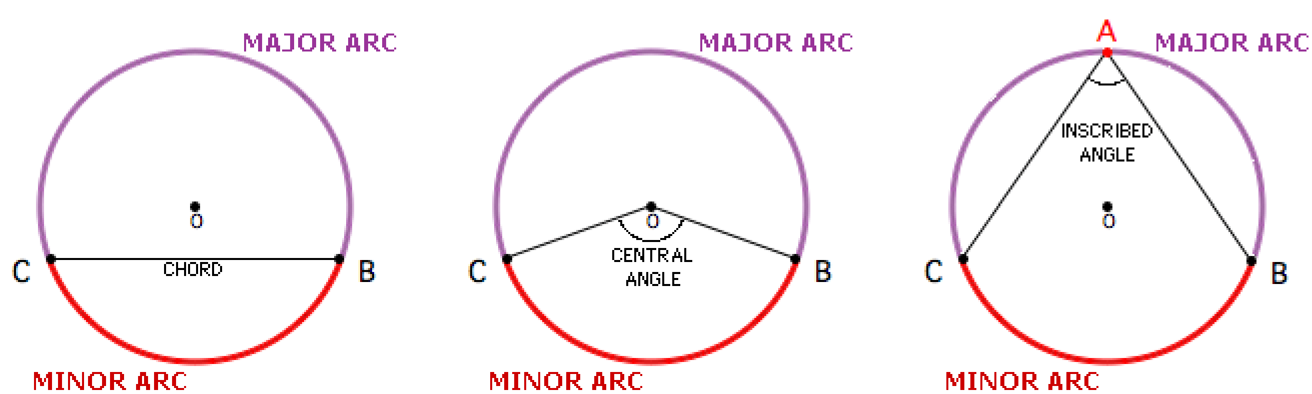
*An inscribed angle is an angle formed by two chords in a circle which have a common endpoint. This common endpoint forms the vertex of the **inscribed angle**. The other two endpoints define what we call an **intercepted arc** on the circle.*

*An **arc** of a circle is a continuous portion of the circle. It consists of two endpoints and all the points on the circle between these endpoints. The ⌒ symbol is used to denote an arc. This symbol is written over the endpoints that form the arc. There are three types of arcs:*

***Semicircle:** an arc whose endpoints are the endpoints of a diameter. It is named using three points. The first and third points are the endpoints of the diameter, and the middle point is any point of the arc between the endpoints.*

***Minor arc:** an arc that is less than a semicircle. A minor arc is named by using only the two endpoints of the arc.*

***Major arc:** an arc that is more than a semicircle. It is named by three points. The first and third are the endpoints, and the middle point is any point on the arc between the endpoints.*



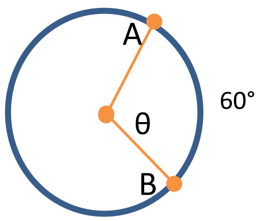
*An **arc** of a circle is a continuous portion of the circle. It consists of two endpoints and all the points on the circle between these endpoints.*

***Arcs** are measured in two ways: as the measure of the Central Angle, or as the length of the arc itself.*

This angle measure is written like this: m AB = 60° and is read as "the measure of arc AB is 60 degrees".

If the minor arc $\overset{\frown}{AB}$ = 60°, then the major arc is equal to 360° - 60°, which is 300°.

When arc angle measures are marked on a diagram, there are two common ways to do it: (1) write the angle alongside the arc itself, or (2) You can draw the lines from the arc endpoints to the center point and label the central angle in the usual way.



*The **arc length** is the measure of the distance along the curved line making up the arc. It is longer than the straight line distance between its endpoints (which would be a chord). The arc length of a circle can be found by first calculating what fraction of the circle is represented by the central angle. Finally, you find the fraction of that circumference (ϑ/360◦). Then you will need to find the circumference of the circle. Circumference is equal to 2πr. Thus the formula for the arc length is:* $\mathbf{C_a = 2\pi r(\vartheta /360)}$*, where* $\mathbf{C_a}$ *is the arc length,* ***r*** *is the radius of the circle and* ***ϑ*** *is the Central Angle of the arc.*

This formula is for using the degree measurement of the Central Angle. If you were using radians, the formula would be: ***R ϑ****, where* ***ϑ*** *is the central angle of the arc in radians and* ***R*** *is the radius of the arc.*

For example, if a circle has a radius of 10 cm and a central angle of 60°, to calculate the arc length for that angle we must first determine what fraction of the circle is represented by the central angle 60°. Since there are 360° in a full circle, we can find the fraction of a circle by simplifying 60/360; 60/360 = 1/6 or 0.16666666666 or 1.67 if we round to the nearest hundredth. So the sector represents 1/6 of the circle.

To find the arc length, we now need to find the circumference of the circle. The Circumference of the circle is equal to 2πr.

C = 2πr
C = 2π x 10
C= 62.8 cm

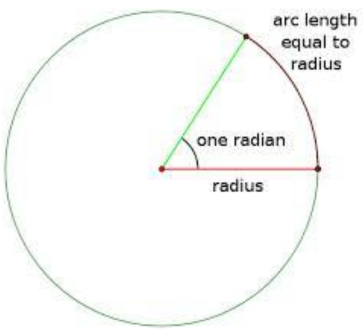
The circumference of this circle is approximately 62.8 cm. To find the arc length, we need to find 1/6 of 62.8 cm.

1/6 x 62.8 = 10.5 cm

Thus the arc length formed by this central angle is approximately 10.5 cm.

To make this process quicker, you can refer to the formula used to determine the arc length formed by a central angle:

$C_a = 2\pi r(\theta/360)$, where C_a is the arc length, **r** is the radius of the circle and **θ** is the Central Angle of the arc.



The **radian** is a unit of measure for angles used mainly in trigonometry. It is used instead of degrees. Whereas a full circle is 360 degrees, a full circle is just over 6 radians. A radian is defined by an arc of a circle. The length of the arc is equal to the radius of the circle. Because of this, the radian is a fixed size no matter what the size of the circle is.
Recall that the circumference of a circle is $2\pi R$, so that means there are 2π, or roughly 6.28 radians in a full circle.
Because a full circle is also exactly 360°, each radian comes out to approximately 57.296°.

Degrees	Radians (exact)	Radians (approx)
30°	π/6	0.524
45°	π/4	0.785
60°	π/3	1.047
90°	π/2	1.571
180°	π	3.142
270°	3π/2	4.712
360°	2π	6.283

Example: How Many Radians in a Full Circle?

Imagine you cut up pieces of string exactly the length from the **center of a circle to its edge** ...... how many pieces do you need to go **around the edge** of the circle?

Answer: **2π** (or about **6.283** pieces of string).

Converting Between Radians and Degrees

Each of radians and degrees has its place. If you're describing directions to me, I'd really rather you said, "Turn sixty degrees to the right when you pass the orange mailbox", rather than, "Turn one-third πradians" at that point. However, if I need to find the area of a sector of a circle, I'd rather you gave me the numerical radian measure that I can plug directly into the formula, rather than the degree measure that I'd have to convert first.

But you won't always be given angle measures in the form you'd prefer, so you'll need to be able to convert between radians and degrees. To do this, you'll use the fact that 360° is "once around", and so is 2π. However, you'll use this fact in the form of the somewhat simplified correspondence of 180° to π.

- **Convert 270° to radians.**

  Since 180° equates to π, then:

$$\frac{270}{1}\times\frac{\pi}{180}=\frac{\cancel{270}^{3}}{1}\times\frac{\pi}{\cancel{180}_{2}}=\frac{3\pi}{2}$$

The equivalent angle is $\frac{3\pi}{2}$

- **Convert $\frac{\pi}{6}$ radians to degrees.**

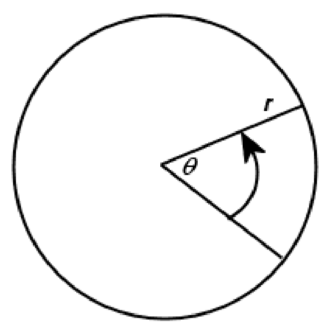
$$\frac{\pi}{6}\times\frac{180}{\pi}=\frac{\cancel{\pi}}{{}_{1}\cancel{6}}\times\frac{\cancel{180}^{30}}{\cancel{\pi}}=30$$

The equivalent angle is 30°

Activity Follow-Up & Extensions:

1. Have Students begin with Lesson Cards 7 – 11 (Radians, Central Angles & Inscribed Angles, Naming Circles, Angles, & Arcs, Arc of a Circle & Arc Length.
2. Have Students begin Command Cards Level Purple (Central Angles & Arc Length).

Presentation III: ***Area of Sectors & Segments***



A Sector of a circle is a pie-shaped region bounded by an arc and an angle. If you know the radius of a circle and a central angle, you can determine the sector area for that angle by first calculating the fraction of the circle that the sector represents, then finding the area of the entire circle, and finally calculating that fraction of the total area.

The area A of a circle with radius r is given by $A = \pi r^2$. The circumference C of that same circle is given by $C = 2\pi r$. But these are the formulas for the entire circle. Sometimes you will need to work with just a portion of a circle's revolution, or with many revolutions of the circle.

If you start with a circle with a marked radius line, and turn the circle a bit, the angle marked off by the original and final locations of the radius line is the "subtended" angle of the "sector", the sector being the pie-wedge-shaped section of the circle. This angle can also be referred to as the "central" angle.

What is the area A of the sector subtended by the marked central angle ϑ? What is the length s of the arc, being the portion of the circumference subtended by this angle?

To determine these values, take a closer look at the area and circumference formulas. The area and circumference are for the entire circle, one full revolution of the radius line. The subtended angle for "one full revolution" is 2π. So the formulas for the whole circle can be restated as:

$$A = \left(\frac{2\pi}{2}\right) r^2$$

$$C = \left(2\pi\right) r$$

For example, if we use a circle with the radius of 10cm and a central angle of 60 °, we can determine the sector area by first finding the fraction of the circle, which we calculated earlier to be 1/6 and then calculating the area of the entire circle. The area of the circle is equal to πr^2. Thus the area of the entire circle is:

$A = \pi r^2$
$A = \pi \times 10^2$
$A = 3.14 \times 100$
$A = 314\ cm^2$

Now we just need to find 1/6 of 314 cm^2 to determine the area of the sector formed by this central angle.

$1/6 \times 314 = 52.3\ cm^2$

The area of the sector formed by the central angle is about 52.3 cm^2.

The formula used to determine the sector area for any central angle is

$A_ș = (\vartheta/360°)\pi r^2$,

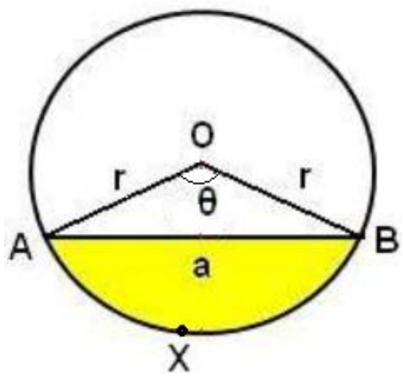
where $A_ș$ is the area of the sector and r is the radius of the circle.

If you do not know the central angle, you can then use the arc length to determine the area of a sector using the formula:

$A_ș = (C_ș/2\pi r)\pi^2$,

*where **Cș** is the arc length and **r** is the radius.*

While a sector looks like a "pie" slice, a **segment** looks like the "pie" slice with the triangular portion cut off. The segment is only the small partially curved figure left when the triangle is removed.



A **chord** of a circle divides the circle into two regions, which are called the **segments** of the circle. The **minor segment** is the region bounded by the chord and the minor arc intercepted by the chord.
The **major segment** is the region bounded by the chord and the major arc intercepted by the chord.

The area of a circle segment is the area of a sector minus the area of the triangle.

To compute the area of a segment, just subtract the area of the triangle from the area of the sector :
Area of a sector: Aș = $(\vartheta/360°)\pi r^2$ ***(if Central Angle is known) or***
Aș = $(Cș/2\pi r)\pi^2$ ***(if Arc Length is known)***
Area of a Triangle: ½bh

Activity Follow-Up & Extensions

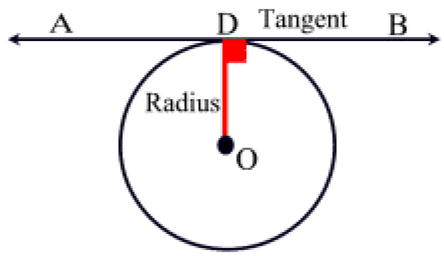
1. Have Students begin with Lesson Cards 12 - 13(Area of a Sector, Area of a Segment)
2. Have Students begin Command Cards Blue (Arc Length, Sectors, & Segments).

Presentation V: Tangents

A **tangent** to a **circle** is perpendicular to the radius at the point of tangency. This is a very useful property when the radius that connects to the point of tangency is part of a right angle, because the trigonometry and the Pythagorean Theorem apply to right triangles.

Theorem 1:

If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of tangency.

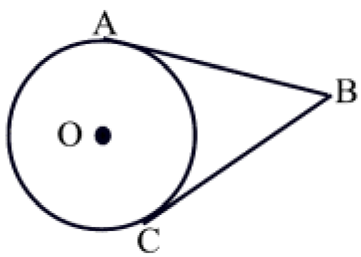


IF: $\overline{AB}$ is a tangent

D is point of tangency

THEN: $\overline{OD} \perp \overline{AB}$

Theorem 2:
Tangent segments to a circle from the same external point are congruent.

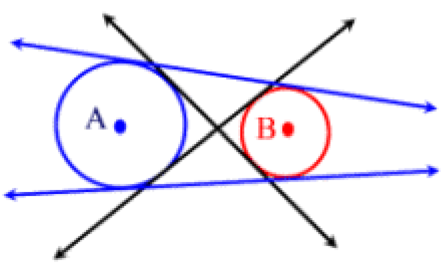


IF: $\overline{AB}$ is a tangent to circle O at A

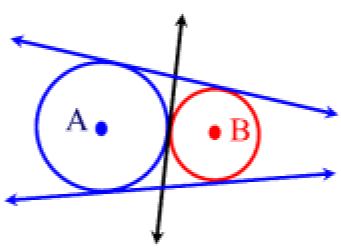
$\overline{CB}$ is a tangent to circle O at C

THEN: $\overline{AB} \cong \overline{CB}$

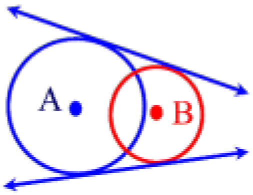
Common tangents are lines or segments that are tangent to more than one circle at the same time.



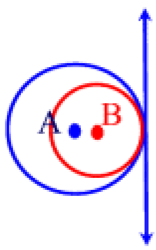
4 Common Tangents
2 external tangents (blue)
2 internal tangents (black)



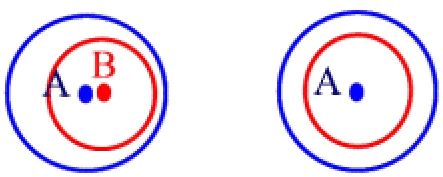
3 Common Tangents
2 external tangents (blue)
1 internal tangent (black)



2 Common Tangents
2 external tangents (blue)
0 internal tangents



1 Common Tangent
1 external tangent (blue)
0 internal tangents



0 Common Tangents
0 external tangents
0 internal tangents

Activity Follow-Up & Extensions:

1. Have Students begin with Lesson Cards 14 (Tangents).